

ABCD of Beta Ensembles and Topological Strings

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Abstract

We study β -ensembles with B_N , C_N , and D_N eigenvalue measure and their relation with refined topological strings. Our results generalize the familiar connections between local topological strings and matrix models leading to A_N measure, and illustrate that all those classical eigenvalue ensembles, and their topological string counterparts, are related one to another via various deformations and specializations, quantum shifts and discrete quotients. We review the solution of the Gaussian models via Macdonald identities, and interpret them as conifold theories. The interpolation between the various models is plainly apparent in this case. For general polynomial potential, we calculate the partition function in the multi-cut phase in a perturbative fashion, beyond tree-level in the large- N limit. The relation to refined topological string orientifolds on the corresponding local geometry is discussed along the way.

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Contents

1	Introduction	2
2	Macdonald ensembles	6
3	Gaussian ensembles: Large N partition functions	8
3.1	Large N expansions	8
3.2	Implications for toric Calabi-Yau backgrounds	13
4	Gaussian ensembles: Correlators	15
5	Multi-cut potentials: Perturbative calculation	17
5.1	Saddle-point approximation	18
5.2	Examples: B_N and D_N quartic	20
6	B-model verification of B_N and D_N quartic	23
6.1	Tree-level geometry	23
6.2	One-loop	26
7	Conclusion	27
	Acknowledgments	28
A	Free energies	29
	References	31

1 Introduction

Eigenvalue ensembles with A_N measure to a power of β ,¹ widely known just as β -ensembles, and their relation to topological gauge and string theories have been studied extensively in recent years. The special instance $\beta = 1$ of generalized interest is the Dijkgraaf-Vafa relation [1] between matrix models, supersymmetric gauge theory and the topological string. In more recent times, the focus has shifted to the more general situation with arbitrary β , which relates the eigenvalue ensembles to Ω -deformed gauge theories, refined topological string theory [2, 3] and the AGT conjecture [4]. Here, the

¹ A_N denotes the finite Coxeter group, and β a positive real number. We review the definitions in section 2.

equivariant parameters ϵ_i of the Ω -deformation, the ensemble parameter β and the string coupling g_s are related via [2],

$$\epsilon_1 = \sqrt{\beta}g_s, \quad \epsilon_2 = -\frac{1}{\sqrt{\beta}}g_s. \quad (1.1)$$

One may note that neither the matrix model nor the topological string at present knows a microscopic interpretation for the deformation parameter β . Rather, the mutual agreement of results calculated with different schemes, the consistency of the space-time interpretation via BPS state/instanton counting, as well as the relation with the Ω -deformed gauge theory, especially in the Nekrasov-Shatashvili limit [5] give confidence that one should view all these models as integral part of a larger interconnected web of theories, thereby in fact defining various notions of quantum geometry, such as that of [3].

The prototypical example for much of this is the Gaussian model, with quadratic potential for the eigenvalues and corresponding, respectively, to a deformed conifold target space, (refined) Chern-Simons theory [6], as well as the $c = 1$ non-critical string at radius $R = \beta$ [7, 2]. This Gaussian model also serves as building block for more general backgrounds.

The purpose of the present paper is to take this logic one step further, and to study the possible role played by eigenvalue ensembles with other finite group measures, specifically, B_N , C_N , and D_N . These models, which we will refer to as Macdonald ensembles, are rather natural, and easily defined, but have been less studied in the recent topological string/gauge theory literature. The ensembles at $\beta = 1$ appeared briefly in the context of the Dijkgraaf-Vafa relation to four-dimensional $\mathcal{N} = 1$ gauge theories with SO/Sp gauge groups and adjoint matter, and the realization of these gauge theories as string theoretic orientifolds. Most closely related to the spirit of the present work are [8] and [9]. Due to the nature of the original DV conjecture, these studies were essentially confined to tree-level. One of the aims of this work is to study the B_N/C_N and D_N eigenvalue ensembles with general β beyond tree-level in greater detail.²

It is then natural to expect that the B_N , and D_N Macdonald ensembles with general $\beta \neq 1$ are related to a refinement of topological string orientifolds, which was one of

²Since the root systems of B_N and C_N differ only in the length of the roots, hence the Haar measures are identical up to an overall factor (see for instance [10]), it will be sufficient for us to consider only the B_N and D_N ensembles.

the original motivations for the present work.³ In thinking about the various pictures, it is however important to remember that these eigenvalue ensembles at $\beta = 1$ are in general *not* identical to the usual SO and Sp matrix models. Rather, the latter models provide the microscopic realization of the A_N ensemble at $\beta = 2$ and $\beta = 1/2$, respectively. They are dual to $\mathcal{N} = 1$ SO/Sp gauge theory with matter in the symmetric/antisymmetric representation. In particular, the orientifold in the large- N dual topological string side acts differently on the tree-level geometry [12, 13].

On the other hand we have the realization, in the Gaussian model, of the β -parameter as the radius of the circle for $c = 1$ non-critical string. There, the orbifold of the $c = 1$ CFT at the self-dual radius is indeed equivalent to the $R = 2$ circle theory. This connection suggests the existence of an *entire new* branch of topological string/matrix model dual pairs that connects up to the standard branch at $\beta = 2$. Our work suggests that this is where the B_N and D_N Macdonald ensembles fit in.

Whereas the duality between $\mathcal{N} = 2$ $U(N)$ gauge theory softly broken to $\mathcal{N} = 1$, A_N eigenvalue ensemble with $\beta = 1$, and topological string theory on the (spectral curve) geometry at large N has been discussed and checked exhaustively in many works, for general β much less is known. For B_N and D_N , even at $\beta = 1$, no higher genus check of the proposed duality between the eigenvalue ensemble and topological string orientifolds has been performed. The power of β plays a major role beyond tree-level, and hence one might hope to be able to learn something about refined topological string theory and orientifolds thereof along the way, which are expected to be related to these β -ensembles in the large N limit.

In fact, under which specific conditions the eigenvalue ensembles for $\beta \neq 1$ relate to refined topological string theory in the large N limit has not been pointed out so far in the literature, even for the ordinary A_N measure. Some examples where such a relation holds in a non-trivial manner were reported in [2, 3]. Although the calculations of [3] were restricted to the cubic ensemble at 1-loop level, there is little doubt that the observed correspondence extends to all genus, at least for the cubic. On the other hand, the Chern-Simons matrix models studied in [14] appear to indicate that in general such a relation does not hold. Attempts to formulate a refined version of the remodeled B-model of [15] have also failed to our knowledge so far.

Some of the problems with the general applicability of the A_N type β -ensemble can

³Meanwhile, the refinement of topological string orientifolds has been studied, with a different perspective and motivation, by Aganagic and Schaeffer [11].

be traced back all the way to tree-level, that is, to the dual spectral curve geometry of the ensemble. This is most clearly visible at hand of the remodeled B-model geometries of [16]: In general, the spectral curve of the eigenvalue ensemble differs from the usual B-model target space geometry of the dual topological string and has singular points. Singularities are a general indication that refinement, *i.e.*, a deformation of the correspondence away from $\beta = 1$, will fail. Indeed, singularities in the B-model geometry could harbor blow-up modes, which spoil an invariant BPS state counting. The corresponding mirror statement is the well-known fact that in order to have a well-defined BPS state counting of left and right spin (and not just the index), the A-model/M-theory geometry should be rigid (*i.e.*, have no complex structure deformations). This leads us to a condition on an A_N type β -ensemble to have a well-defined BPS state counting interpretation. Namely, the spectral curve has to be non-singular. In particular, this applies as well to ensembles with polynomial potentials, *i.e.*, one has to fill all cuts to ensure that one has a well defined BPS index. Under this restriction, the duality of [2] has a chance to survive the β -deformation in a quite general setting. Similar considerations apply to the B_N and D_N cases, up to some technicality which we will explain in more detail in section 5.1. Confirmation for this expectation will be found at hand of A_N , B_N and D_N β -ensembles with quartic potential, which appear to be as well compatible with a (refined) topological string interpretation, as the free energies fulfill the 1-loop holomorphic anomaly equation.

The outline is as follows. In section 2 we will give the definition of Macdonald ensembles with special emphasize on A_N , B_N and D_N . This is followed by a detailed discussion of the large N expansion of the Gaussian partition functions and implications thereof for refined topological string orientifolds, in section 3. In section 4 a recursion relation satisfied by Gaussian correlators is derived (generalizing [17, 18]), which constitute an essential ingredient for the explicit calculation of the multi-cut ensemble partition function, which section 5 is about. In subsection 5.1, we will give a generalization of the framework of [19, 20, 17, 18] to B_N/D_N , and apply it in section 5.2 to the model with quartic potential. The B-model verification of the tree-level and 1-loop results of section 5.2 will be performed in section 6. We conclude in section 7. In appendix A the explicit results for the g_s expansion of the free energy of the B_N and D_N β -ensemble with quartic potential are attached.

2 Macdonald ensembles

Let \mathcal{G} be a finite group of isometries of \mathbb{R}^N generated by reflections in hyperplanes through the origin (*i.e.*, a finite reflection or Coxeter group). Let there be h hyperplanes, each defined by a condition on $\lambda \in \mathbb{R}^N$ of the form

$$\sum_{i=1}^N a_{\alpha,i} \lambda_i = 0,$$

where $a_\alpha \in \mathbb{R}^N$, $\alpha = 1, \dots, h$. The group \mathcal{G} naturally acts on the algebra of polynomial functions on \mathbb{R}^N . The \mathcal{G} -invariant polynomials form an \mathbb{R} -algebra generated by N independent polynomials of degrees d_i , $i = 1, \dots, N$. Normalizing the vectors $(a_{\alpha,i})_i$ via $\sum_{i=1}^N a_{\alpha,i}^2 = 2$, define the particular invariant polynomial

$$P_{\mathcal{G}}(\lambda) = \prod_{\alpha=1}^h \sum_{i=1}^N a_{\alpha,i} \lambda_i.$$

Macdonald conjectured the integral identity [21]

$$Z_{\mathcal{G}}(\beta) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} [d\lambda] |P_{\mathcal{G}}(\lambda)|^{2\beta} e^{-\frac{1}{2} \sum_{i=1}^N \lambda_i^2} = \prod_{i=1}^N \frac{\Gamma(1 + d_i \beta)}{\Gamma(1 + \beta)}, \quad (2.1)$$

where $[d\lambda] := \prod_{i=1}^N d\lambda_i$, $\beta \in \mathbb{C}$ with $\operatorname{Re} \beta > 0$. A proof of this identity has been given by Opdam [22, 23]. $Z_{\mathcal{G}}(\beta)$ is also referred to as Macdonald integral.

For $\mathcal{G} = A_{N-1}$ the Macdonald integral specializes to Mehta's integral. For this work, in addition the cases $\mathcal{G} = B_N$ and $\mathcal{G} = D_N$ are of particular interest. These give rise to the integral identities,

$$\begin{aligned} Z_{A_{N-1}}(\beta) &:= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} [d\lambda] \Delta(\lambda)^{2\beta} e^{-\frac{1}{2} \sum_i \lambda_i^2} = \prod_{i=1}^N \frac{\Gamma(1 + i\beta)}{\Gamma(1 + \beta)}, \\ Z_{B_N}(\beta) &:= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} [d\lambda] \Delta(\lambda^2)^{2\beta} \prod_{i=1}^N \lambda_i^{2\beta} e^{-\frac{1}{2} \sum_i \lambda_i^2} = \prod_{i=1}^N \frac{\Gamma(1 + 2i\beta)}{\Gamma(1 + \beta)}, \\ Z_{D_N}(\beta) &:= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} [d\lambda] \Delta(\lambda^2)^{2\beta} e^{-\frac{1}{2} \sum_i \lambda_i^2} = \frac{\Gamma(1 + N\beta)}{\Gamma(1 + \beta)} \prod_{i=1}^{N-1} \frac{\Gamma(1 + 2i\beta)}{\Gamma(1 + \beta)}, \end{aligned} \quad (2.2)$$

where $\Delta(\lambda)$ denotes the usual Vandermonde determinant, $\Delta(\lambda) := \prod_{i < j} (\lambda_i - \lambda_j)$. We are particularly interested in the large N limit thereof, see section 3.

Viewing the above integrals as partition functions of Gaussian eigenvalue ensembles, it is natural to define general Macdonald ensembles by replacing the quadratic term

$\sum_i \lambda_i^2$ with a general “single-trace” polynomial potential $\sum_i W(\lambda_i)$. For $\mathcal{G} = A_{N-1}$ the Macdonald ensemble is identical to the usual β -ensemble. For $\mathcal{G} = B_N$ and D_N , and with $\beta = 1$, these ensembles correspond to the ones considered in [8] in the context of the Dijkgraaf-Vafa relation (with the additional condition $W(x) = W(-x)$).

It is convenient to parameterize the measure $P_{(b,d)}(\lambda) := P_{\mathcal{G}}(\lambda)$ with \mathcal{G} being A_N , B_N or D_N as

$$P_{(b,d)}(\lambda) = \Delta_+(\lambda)^{b+d} \Delta_-(\lambda) \prod_{i=1}^N \lambda_i^b, \quad (2.3)$$

where we defined

$$\Delta_{\pm}(\lambda) := \prod_{i < j}^N (\lambda_i \pm \lambda_j).$$

In particular $\Delta_-(\lambda) = \Delta(\lambda)$, corresponds to the usual Vandermonde, and $\Delta_-(\lambda)\Delta_+(\lambda) = \Delta(\lambda^2)$. For $(b,d) = (0,0)$ we get $P_{A_N}(\lambda)$, $(1,0)$ yields $P_{B_N}(\lambda)$ and $(0,1)$ results in $P_{D_N}(\lambda)$. Hence the Macdonald ensembles $Z_{\mathcal{G}}(\beta)$ with $\mathcal{G} = A_N$, $\mathcal{G} = B_N$ or $\mathcal{G} = D_N$ can be treated simultaneously via the ensemble ⁴

$$Z_{(b,d)}(\beta) \sim \int [d\lambda] |P_{(b,d)}(\lambda)|^{2\beta} e^{-\sum_{i=1}^N W(\lambda_i)}, \quad (2.4)$$

with $P_{(b,d)}(\lambda)$ as defined in (2.3).

The expectation value for an operator insertion $\hat{\mathcal{O}}$ is defined as usual as

$$\langle \hat{\mathcal{O}} \rangle_{(b,d)} := \int [d\lambda] |P_{(b,d)}(\lambda)|^{2\beta} \hat{\mathcal{O}} e^{-\sum_{i=1}^N W(\lambda_i)}.$$

Trivially, we have $\langle 1 \rangle_{(b,d)} = Z_{(b,d)}(\beta)$.

It is instructive to compare the formulas for B_N and D_N . The only difference is the additional factor of $\prod_{i=1}^N \lambda_i^{2\beta}$ for B_N , and can be interpreted as follows. We know from the A_N β -ensemble that the insertion of a brane at position x in the dual geometry corresponds to the insertion of some power of a determinant factor of $\prod_{i=1}^N (x - \lambda_i)$ times an overall classical piece of $\psi_{cl}(x) = e^{W(x)}$ [24, 3]. Different powers of the insertion correspond to different types of branes [3]. Thus, the insertion of a brane at x plus a mirror brane at $-x$ corresponds to an operator insertion of

$$\hat{\Psi}_{\beta}(x) \hat{\Psi}_{\beta}(-x) = \psi_{cl}(x) \psi_{cl}(-x) \prod_{i=1}^N (\lambda_i^2 - x^2)^{\beta}. \quad (2.5)$$

⁴The g_s dependence needed to match to topological strings will be brought in via $W(x)$ and, if necessary, a rescaling of the eigenvalues.

This implies that the B_N ensemble can be understood as the D_N ensemble with insertion of an additional pair of branes at the origin ($x = 0$), *i.e.*,

$$Z_{(1,0)}(\beta) \sim \langle \hat{\Psi}_\beta(0) \hat{\Psi}_\beta(0) \rangle_{(0,1)} = Z_{(0,1)}(\beta) \Psi_{(0,1)}^{2\beta}(0), \quad (2.6)$$

where we defined the partition function with h coincident (β -) branes $\Psi_{(b,d)}^{h\beta}(x)$ in the background parameterized by (b, d) as

$$\Psi_{(b,d)}^{h\beta}(x) := \frac{\langle (\hat{\Psi}_\beta(x))^h \rangle_{(b,d)}}{\langle 1 \rangle_{(b,d)}}. \quad (2.7)$$

3 Gaussian ensembles: Large N partition functions

In this section, we review in some detail the large- N expansions of the Gaussian partition functions (2.2), as well as the various ways that these enter into the topological string.

3.1 Large N expansions

A_N

In contrast to the B_N and D_N ensembles, the 't Hooft large- N limit of $Z_{A_{N-1}}(\beta)$ has been studied extensively in the physics literature. The asymptotic expansion as $N \rightarrow \infty$ is related to the “Schwinger” integral,

$$\log Z_{A_{N-1}}(\beta) \sim \int \frac{dt}{t} \frac{e^{-\mu t}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)} =: \log Z_A(g_s, \beta), \quad (3.1)$$

where ϵ_1, ϵ_2 are related to g_s, β as in (1.1), and we have

$$\mu := \sqrt{\beta} g_s N. \quad (3.2)$$

We note the obvious symmetry of the partition function under $\epsilon_1 \leftrightarrow \epsilon_2$. As $g_s \rightarrow 0$, we have the well-known asymptotic expansion

$$\int \frac{dt}{t} \frac{e^{-\mu t}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)} \sim \sum_{n=0}^{\infty} \Phi_A^{(n)}(\beta) \left(\frac{g_s}{\mu} \right)^n, \quad (3.3)$$

with certain polynomial expressions $\Phi_A^{(n)}(\beta)$.⁵

⁵Really, $\beta^n \Phi_A^{(n)}$ is a polynomial in β .

As is well-known, for n even, the $\Phi_A^{(n)}(\beta)$ specialize at $\beta = 1$ to give the virtual Euler characteristic of the moduli space \mathcal{M}_g of genus $g = \frac{n}{2} + 1$ complex curves [25],

$$\Phi_A^{(2g-2)}(1) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g-2)}, \quad (3.4)$$

where B_n are the Bernoulli numbers. For n odd, the $\Phi_A^{(n)}(1)$ vanish.

On the other hand, for n odd, the $\Phi_A^{(n)}(\beta)$ specialize at $\beta = 2$ to give the virtual Euler characteristic of the moduli space $\mathcal{M}_{\tilde{g}}^O$ of complex curves of genus $\tilde{g} = n + 1$ with a fixed-point free anti-holomorphic involution (*i.e.*, certain type of real curves) [26, 27, 28], up to a rescaling of g_s . In string theory language, the quotients give unoriented Riemann surfaces with genus $g = \tilde{g}/2$, (\tilde{g} being even), no boundaries, and one crosscap,

$$\Phi_A^{(2g-1)}(2) = -2^{1/2-g} \chi(\mathcal{M}_{\tilde{g}}^O) = -2^{1/2-g} \frac{(2^{2g-2} - 2^{-1})B_{2g}}{2g(2g-1)}. \quad (3.5)$$

For n even, we have that $\Phi_A^{(2g-2)}(2) = 2^{-g} \chi(\mathcal{M}_g)$. Hence, the coefficients at $\beta = 2$ show the typical structure of an orientifold

$$2^{n/2} \Phi_A^{(n)}(2) = \frac{1}{2} \chi(\mathcal{M}_g) - \chi(\mathcal{M}_{\tilde{g}}^O). \quad (3.6)$$

For later reference, note that via making use of (3.1) and (3.6), one can infer as well an integral representation of the generating function for the $\chi(\mathcal{M}_{\tilde{g}}^O)$, *i.e.*,

$$\mathcal{T}(g_s) := \log \frac{Z_A(\sqrt{2}g_s, 2)}{\sqrt{Z_A(g_s, 1)}} = -\frac{1}{2} \int \frac{dt}{t} \frac{e^{-\mu t}}{e^{g_s t} - e^{-g_s t}} \sim \sum_{n=0}^{\infty} \chi(\mathcal{M}_{2n}^O) \left(\frac{g_s}{\mu} \right)^{2n-1}. \quad (3.7)$$

A similar, and in fact related, “Schwinger” integral has appeared before in the context of the orientifold constant map contribution [29, 30].

The fact that the A_N β -ensemble can be used to interpolate between the Euler characteristic of moduli spaces of complex and real curves was pointed out in [27], and interpreted as a geometric parameterization. In particular, it was conjectured that the $\Phi_A^{(n)}(\beta)$ themselves should describe the Euler characteristic of some related moduli space.

Although the appearance of the moduli of real curves is suggestive, a simple closed string theory interpretation is hampered by the fact that the expansion (3.3) for $\beta \neq 1$ contains terms of both even and odd powers of g_s . This originates from the fact that

(3.1) is not invariant under $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$, except when $\epsilon_1 = -\epsilon_2$. As is by now well-appreciated, the additional (quantum) shift

$$\mu \rightarrow \mu + \frac{\epsilon_1 + \epsilon_2}{2} \quad (3.8)$$

restores that symmetry. We have the asymptotic expansion

$$\int \frac{dt}{t} \frac{e^{-\mu t}}{(e^{\epsilon_1 t/2} - e^{-\epsilon_1 t/2})(e^{\epsilon_2 t/2} - e^{-\epsilon_2 t/2})} \sim \sum_n \Psi_A^{(n)}(\beta) \left(\frac{g_s}{\mu}\right)^n \quad (3.9)$$

with $\Psi_A^{(n)}(\beta) \equiv 0$ for n odd. This shifted partition function is also identical to the partition function of the $c = 1$ string at radius $R \propto \beta$, originally found in [31]. From the above formulas it is clear that $\Phi_A^{(n)}(1) = \Psi_A^{(n)}(1)$.

Turning to the topological string, it was discovered long time ago in [7], that the integral (3.1) at $\beta = 1$, *i.e.*, the $c = 1$ string at the self-dual radius, governs the leading behavior of the B-model topological string in the limit in which the target space develops a conifold singularity, as the complex structure parameter $\mu \rightarrow 0$. As explained for instance in [32], the coefficients $\Phi_A^{(n)}(1) = \Psi_A^{(n)}(1)$ therefore provide universal boundary condition for solving the topological string via holomorphic anomaly equation [33]. As shown in [34, 35], see also [36], the one-parameter deformation $\Psi_A^{(n)}(\beta)$ provides the analogous boundary conditions for solving the refined topological string in the B-model via the same holomorphic anomaly equation. (Alternatively, one may use the extended holomorphic anomaly equation of [37] with boundary conditions provided by the $\Phi_A^{(n)}(\beta)$ to solve for the refined topological string amplitudes after undoing the quantum shift.) This observation confirms the identification of β as the radius R of $c = 1$ string [2].

A seemingly unrelated observation is the fact that the coefficients $\Psi_A^{(n)}(2)$ also have a topological string interpretation, in the context of the real topological string [38]. Namely, writing [34],

$$2^{n/2} \Psi_A^{(n)}(2) = \frac{1}{2} (\Psi_A^{(n)}(1) + \Psi_{\text{KB}}), \quad (3.10)$$

the Ψ_{KB} control the leading behavior of the topological string amplitude on a genus g Klein-bottle (an unoriented Riemann surface with genus g and even number of cross-caps) around a conifold point in moduli space. This relation begs for a topological interpretation of the Ψ_{KB} similar to that of the $\Psi_A^{(n)}(1)$ in terms of the moduli space of genus g complex curves (with $n = 2g - 2$). We note that whatever this interpretation is, $\Psi_A^{(n)}(1)$ is *not* the virtual Euler characteristic of moduli of complex curves

with fixed-point free anti-holomorphic involution of odd genus $\tilde{g} = 2g - 1$ (which are the covers of these higher genus Klein bottles) studied in [27], which vanishes, but should be closely related to it. It is also interesting to note that the quantum shift (3.8) transforms Klein-bottle contributions into cross-cap contributions, as is apparent via comparing (3.6) and (3.10).

B_N

It follows from (2.2) that $Z_{B_N}(\beta)$ can be expressed in terms of $Z_{A_{N-1}}(\beta)$ as

$$\log Z_{B_N}(\beta) = \log Z_{A_{N-1}}(2\beta) + N \log \frac{\Gamma(1 + 2\beta)}{\Gamma(1 + \beta)}.$$

Since in our 't Hooft limit, we neglect the most singular terms, of positive power of N , we simply write,

$$Z_B(g_s, \beta) \sim Z_A(g_s, 2\beta). \quad (3.11)$$

As a result, the $\beta \leftrightarrow 1/\beta$ symmetry of $Z_A(\beta)$ translates to a $\beta \leftrightarrow 1/(4\beta)$ symmetry of the $Z_B(\beta)$ partition function. Similarly as in the A_N case, we denote the expansion coefficients of the g_s expansion of the corresponding (shifted) free energy as $\Psi_B^{(n)}(\beta)$. Obviously,

$$\Psi_B^{(n)}(\beta) = \Psi_A^{(n)}(2\beta).$$

For later reference, let us explicitly state the “1-loop” coefficient which the (shifted) $Z_B(g_s, \beta)$ implies, *i.e.*,

$$\Psi_B^{(0)}(\beta) = \frac{1}{48} \left(\frac{1}{\beta} + 4\beta \right). \quad (3.12)$$

D_N

For the D_N Macdonald ensemble, we sort terms such that

$$\log Z_{D_N}(\beta/2) = \log Z_{A_{N-1}}(\beta) - \log \frac{\Gamma(1 + N\beta)}{\Gamma(1 + N\beta/2)} + N \log \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta/2)}. \quad (3.13)$$

Thus, besides the A_N term, we have one additional N dependent term which we don't neglect in the large- N limit. Invoking the integral representation of the digamma function $\gamma(x) = \frac{d}{dx} \log(\Gamma(x))$,

$$\gamma(x) = \int_0^\infty dt \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right),$$

one can infer that the essential part of the new contribution reads

$$- \int \frac{dt}{t} \frac{e^{-N\beta t} - e^{-N\beta t/2}}{1 - e^{-t}}.$$

Redefining $t \rightarrow t \frac{g_s}{\sqrt{\beta}}$ and taking $N \rightarrow \infty$ while keeping $\sqrt{\beta} N g_s =: \mu$ fixed yields

$$- \int \frac{dt}{t} \frac{e^{-\mu t} - e^{-\mu t/2}}{1 - e^{-\frac{g_s}{\sqrt{\beta}} t}}.$$

Reverting to our usual notation, we obtain (up to non-universal terms)

$$\log Z_{D_N}(\beta/2) \sim \int \frac{dt}{t} \frac{e^{-\mu t}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)} + \int \frac{dt}{t} \frac{e^{-\mu t}}{e^{\epsilon_2 t} - e^{-\epsilon_2 t}} =: \log Z_D(g_s, \beta/2). \quad (3.14)$$

We recognize the second term as being essentially the generating function (3.7). Hence, $Z_D(g_s, \beta)$ is entirely given by a combination of Z_A , *i.e.*,

$$\log Z_D(g_s, \beta) = \log Z_A(g_s, 2\beta) - 2 \mathcal{T}(\sqrt{2\beta} g_s). \quad (3.15)$$

(where \mathcal{T} is defined in (3.7).) It is also instructive to express the “1-loop” coefficient $\Phi_D^{(0)}(\beta)$ contained in (3.14) in terms of $\Phi_A^{(0)}(\beta)$. Using the relations (3.15) and (3.7), we infer

$$\Phi_D^{(0)}(\beta) = \Phi_A^{(0)}(2\beta) + 2 \left(\Phi_A^{(0)}(2) - \frac{1}{2} \Phi_A^{(0)}(1) \right). \quad (3.16)$$

We observe that this result matches the structure of the orbifold branch partition function for the $c = 1$ CFT on the torus derived in [39]. We take this as a hint that (3.15) is related to the partition function of the orbifold branch of the $c = 1$ non-critical string. More precisely, such a relation should hold after an appropriate quantum shift. One way to identify the appropriate D_N -analog of (3.8) is to impose a symmetry under $g_s \rightarrow -g_s$. The symmetry can be motivated as follows. We know that for integer values of β the partition function of the $c = 1$ string on the circle branch can be matched to the partition function of the topological string expanded near a $A_{\beta-1}$ type singularity (under appropriate choice of deformation parameters) [40]. Since the chiral ground ring manifold of the $c=1$ string on the orbifold branch corresponds to a Kleinian singularity of D -type [41], we expect that similarly the orbifold branch partition function can be matched to the topological string expanded near a D -type singularity, implying the symmetry under $g_s \rightarrow -g_s$.

Indeed, after

$$\mu \rightarrow \mu - \frac{\epsilon_2}{2}, \quad (3.17)$$

the free energy (3.14) becomes

$$\int \frac{dt}{t} \frac{e^{-\mu t} \cosh\left(\frac{(\epsilon_1 + \epsilon_2)t}{2}\right)}{2 \sinh(\epsilon_1 t/2) \sinh(\epsilon_2 t)} \sim \sum_n \Psi_D^{(n)}(\beta/2) \left(\frac{g_s}{\mu}\right)^n, \quad (3.18)$$

and clearly possesses an even power only expansion in g_s .⁶ We denote the expansion coefficients by $\Psi_D^{(n)}(\beta)$. For the special value $\beta = 1$, we find in addition

$$\Psi_D^{(n)}(1) = \Psi_B^{(n)}(1) = \Psi_A^{(n)}(2). \quad (3.19)$$

This identity is precisely the one expected from the matching of circle and orbifold branches of $c = 1$ string at $R = 1$ and $R = 2$, respectively. It is important to note however that generally, Z_{A_N} and Z_{D_N} are not related by a simple shift of N .

For later reference, we explicitly state the “1-loop” coefficient

$$\Psi_D^{(0)}(\beta) = \frac{1}{48} \left(\frac{1}{\beta} - 8\beta \right) + \frac{1}{4}. \quad (3.20)$$

3.2 Implications for toric Calabi-Yau backgrounds

The free energy $\mathcal{F}_A(Q; \beta)$ of the refined topological string on the *resolved* conifold geometry, *i.e.*, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$, is related to the refined *deformed* conifold free energy given by the integral (3.9) by identifying the Kähler parameter as $Q = e^{-\mu}$ and simply replacing the integral by a sum

$$\int dt \rightarrow \sum_{d=1}^{\infty}. \quad (3.21)$$

This replacement originates from the sum over states of D0-brane charge $k \in \mathbb{Z}$ and mass $\sim \mu + 2\pi i k$, or, in M-theory language, from the extra state degeneracy due to momenta around the M-theory circle. It is natural to assume that a similar “quantization” as in (3.21) can be applied to the other Macdonald integrals as well. For B_N , we infer from (3.11)

$$\mathcal{F}_B(Q; \beta) = \mathcal{F}_A(Q; 2\beta), \quad (3.22)$$

i.e., the resolved conifold free energy of B_N type agrees with that of type A_N .

⁶It is interesting to note that this generating function looks very similar to the generating function for the massless hypermultiplet contribution occurring in SU(2) gauge theory on Ω -deformed A_1 ALE space [42].

The D_N ensemble is more interesting. Applying (3.21) to the (shifted) D_N free energy (3.18), we obtain

$$\mathcal{F}_D(Q; \beta/2) = \sum_{d=1}^{\infty} Q^d \frac{(q/t)^{d/2} + (q/t)^{-d/2}}{d(q^{d/2} - q^{-d/2})(t^{-d} - t^d)}, \quad (3.23)$$

with the usual definitions $q := e^{\sqrt{\beta}g_s}$ and $t := e^{\frac{g_s}{\sqrt{\beta}}}$, as prediction for the refined free energy of D_N type of the resolved conifold geometry. In particular, we have an even power only expansion in g_s and the relation $\mathcal{F}_D(Q; 1) = \mathcal{F}_A(Q; 2)$ holds.

It is instructive to compare this result with the expectations based on a topological string orientifold interpretation. A convenient reference is the recent proposal [11] for the orientifolded and refined resolved conifold free energy. This proposal, which is obtained from an $SO(2N)$ refined Chern-Simons/geometric transition point of view, reads,

$$\mathcal{F}_A(t^{-1/2}Q; \sqrt{2}g_s, 2\beta) + \sum_{d=1}^{\infty} \frac{((q/t)Q)^{d/2}}{t^{d/2} - t^{-d/2}}. \quad (3.24)$$

(we have here exchanged $q \leftrightarrow t$ (corresponding to $\beta \leftrightarrow 1/\beta$).) As observed in [11], the specialization of (3.24) to $\beta = 1$ equals the free energy of an orientifold of the resolved conifold (acting either in fixed-point free [43], or in a real [44, 29] fashion). The second term in (3.24) can be understood as originating from the second term in (3.7) of the (unshifted) D_N free energy, summing only over even D0 brane charge (up to a shift). It may also be seen as a brane placed at $-1/2 \log Q$ in the A-model geometry. Since the brane is localized in two space-time dimensions, it is exposed only to a single parameter of the Ω -deformation (after a suitable redefinition of parameters).

The structure of the refined orientifold free energy (3.24) is consistent with the results of [34], where it was found that the free energy of the fixed-point free orientifold of $\mathcal{O}(-2) \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ equals the refined free energy at $\beta = 2$ (for this orientifold one has no open string sector).

We draw attention to the fact that, at $\beta = 1$, the shift of $\log Q$ in the first term of (3.24) cancels the part of the sum in the second term coming from even d . That summation would be unusual for the g_s -odd sector of an orientifold. The cancellation is possible because the open string contribution is essentially a closed string period. On the other hand, this kind of comparison challenges the extrapolation of the proposed orientifold structure (3.24) to more general toric Calabi-Yau geometries, since a cancellation between a shift and an open string contribution is not possible in general.

Instead, we propose to view $\mathcal{F}_D(Q; \beta)$ of (3.23) as the refined free energy of type D_N of the resolved conifold, independent of an orientifold interpretation.⁷ Although for specific situations (such as the conifold or Dijkgraaf-Vafa type geometries, see section 5), when the orientifold contribution is a closed period, the (unshifted) D_N free energy can be matched and interpreted at $\beta = 1$ as an orientifold, this relation is not expected to persist in general.

Specifically, we propose to identify the theory of D_N type with the orbifold branch in the $c = 1$ moduli space, extending the identification of A_N with the circle branch of $c = 1$ [2]. The relation $\beta = R$ is the same on both branches. For a toric Calabi-Yau manifold, one may then use (3.18) as boundary condition on the holomorphic anomaly equation at the conifold point in moduli space in order to obtain predictions for the D_N theory. We will not pursue this quite interesting toric direction further in this work, but rather stick to the Macdonald ensemble setting, where the correspondence with orientifolds holds provided we work with even potentials. This will give further evidence via explicit calculations that the deformation (2.4) away from $\beta = 1$ is consistent.

4 Gaussian ensembles: Correlators

In order to evaluate perturbatively β -ensembles with multi-cut support we will need to evaluate normalized Gaussian correlators defined as

$$C_{k_1, k_2, \dots, k_m}^{(b, d)}(\beta) := \frac{\langle \prod_{i=1}^m S_{k_i} \rangle_{(b, d)}}{\langle 1 \rangle_{(b, d)}} = \frac{1}{Z_{(b, d)}(\beta)} \int [d\lambda] |P_{(b, d)}(\lambda)|^{2\beta} e^{-\frac{1}{2} \sum_{i=1}^N \lambda_i^2} \prod_{i=1}^m S_{k_i}, \quad (4.1)$$

with $S_k := \sum_{i=1}^N \lambda_i^k$ and normalized via the Gaussian partition function $Z_{(b, d)}$ given in (2.1). Clearly,

$$C_{0, 0, \dots, 0}^{(b, d)}(\beta) = N \times N \times \dots \times N.$$

Correlators with non-vanishing k_i can be solved for recursively invoking the Ward-identities resulting from invariance under eigenvalue reparameterizations. For example, more recently this approach has been followed in the A_N case with general β in [18]. The generalization of this approach to the B_N and D_N case we are interested in is

⁷It is conceivable that this result (instead of (3.24)) can be obtained also from the Chern-Simons point of view by appropriately incorporating the quantum shift (3.17) in the large- N limit.

straight-forward. The Ward identities read

$$\sum_{k=1}^N \int [d\lambda] \partial_{\lambda_k} \left(\lambda_k^n \left(\Delta_+(\lambda)^{b+d} \Delta_-(\lambda) \prod_{i=1}^N \lambda_i^b \right)^{2\beta} e^{-\frac{1}{2} \sum_{i=1}^N \lambda_i^2} S_{k_1} \dots S_{k_m} \right) = 0. \quad (4.2)$$

Acting with the derivative on each factor and expressing the resulting terms through the correlators (4.1) yields a recursive equation for them. For brevity, we will here explicitly state only the new contributions of $\Delta_+(\lambda)^{b+d}$ and $\prod_{i=1}^N \lambda_i^b$ which do not occur in the A_N case. The remaining A_N contributions can be deduced similarly.

Using,

$$\sum_{k=1}^N \lambda_k^n \partial_{\lambda_k} \left(\prod_{i=1}^N \lambda_i^b \right)^{2\beta} = 2\beta b S_{n-1} \left(\prod_{i=1}^N \lambda_i^b \right)^{2\beta},$$

the new contribution which only occurs for B_N can be inferred to be simply given by

$$2\beta b C_{n-1, k_1, \dots, k_m}^{(b, d)}.$$

The derivation of the contribution of $\Delta_+(\lambda)^{b+d}$ goes as follows. One rewrites

$$\sum_{k=1}^N \lambda_k^n \partial_{\lambda_k} (\Delta_+(\lambda)^{2\beta(b+d)}) = 2\beta(b+d) \Delta_+(\lambda)^{2\beta(b+d)} \sum_{k=1}^N \lambda_k^n \partial_{\lambda_k} \log \Delta_+(\lambda),$$

and deduces the identity

$$2\beta(b+d) \sum_{k=1}^N \lambda_k^n \partial_{\lambda_k} \log \Delta_+(\lambda) = (b+d) \beta \sum_{i \neq j} \frac{\lambda_i^n + \lambda_j^n}{\lambda_i + \lambda_j} = (b+d) \beta \sum_{i \neq j} \frac{\lambda_i^n - (-\lambda_j)^n}{\lambda_i - (-\lambda_j)}.$$

Note that the second step in the above equality is only valid for n odd. Fortunately, knowing only how to deal with the n odd case is sufficient for our purposes. Finally, making use of the identity

$$\sum_{i \neq j} \frac{\lambda_i^n - (-\lambda_j)^n}{\lambda_i - (-\lambda_j)} = \sum_{k=0}^{n-1} \sum_{i \neq j} (-1)^k \lambda_i^{n-k-1} \lambda_j^k = \sum_{k=0}^{n-1} (-1)^k S_{n-k-1} S_k - S_{n-1},$$

one arrives at the contribution

$$\beta(b+d) \sum_{k=0}^{n-1} (-1)^k C_{n-k-1, k, k_1, \dots, k_m}^{(b, d)} - \beta(b+d) C_{n-1, k_1, \dots, k_m}^{(b, d)}.$$

Combining all terms, taking also the usual A_N contribution into account (see for instance [18, 45]), the Ward identities (4.2) translate to the recursive relation

$$C_{n+1,k_1,k_2,\dots,k_m}^{(b,d)} = ((1-\beta)n + (b-d)\beta)C_{n-1,k_1,\dots,k_m}^{(b,d)} + \beta \sum_{k=0}^{n-1} (1 + (b+d)(-1)^k) C_{n-k-1,k,k_1,\dots,k_m}^{(b,d)} + \sum_{j=1}^m k_j C_{k_1,\dots,k_j+n-1,\dots,k_m}^{(b,d)}. \quad (4.3)$$

Note that the sole difference between the B_N and D_N case is a switch of sign of one of the terms entering the recursive relation.

As pointed out already above, in case of B_N and D_N the recursive relation is only valid for n odd. However, the relation closes if furthermore all k_i are even. Hence, it can be used to determine all correlators $C_{k_1,k_2,\dots,k_m}^{(b,d)}$ with k_i even. A few examples follow.

$$\begin{aligned} C_2^{(0,1)} &= N(1 + 2\beta(N-1)), & C_2^{(1,0)} &= N(1 + 2\beta N), \\ C_{2,2}^{(0,1)} &= C_2^{(0,1)}(2 + C_2^{(0,1)}), & C_{2,2}^{(1,0)} &= C_2^{(1,0)}(2 + C_2^{(1,0)}), \\ C_4^{(0,1)} &= C_2^{(0,1)}(3 + 4\beta(N-1)), & C_4^{(1,0)} &= C_2^{(1,0)}(3 + \beta(4N-2)). \end{aligned} \quad (4.4)$$

One should note that while in the A_N case with $\beta = 1$ a generating function for the 1-point correlators $C_n^{(0,0)}$ is known [25] and there are also closed expressions for $\beta = 1/2$ and $\beta = 2$ [46], no such closed formula has been found for general β , nor for the B_N and D_N cases, so far. Nevertheless, we can make at least one general observation regarding the structure of the $C_n^{(b,d)}$ for n even. Namely, the coefficients of the highest powers in N appear to be always given in terms of the Catalan-numbers $C_n := \frac{(2n)!}{(n+1)!n!}$, *i.e.*,

$$\begin{aligned} C_{2n}^{(0,0)}(\beta) &= C_n \beta^n N^{n+1} + \dots, & C_{2n}^{(1,0)}(\beta) &= C_n \beta^n N^{n+1} + \dots, \\ C_{2n}^{(0,1)}(\beta) &= 2^n C_n \beta^n N^{n+1} + \dots. \end{aligned} \quad (4.5)$$

5 Multi-cut potentials: Perturbative calculation

In this section we shall evaluate the eigenvalue ensembles (2.4) in a perturbative fashion via a saddle-point approximation, making use of the fact that for $g_s \rightarrow 0$ the eigenvalues λ_i localize to the critical points of the potential $W(x)$. A detailed exposition of the perturbative calculation of the β -deformed A_N partition function has been given in [18], as a generalization of the earlier $\beta = 1$ works [19, 20, 17]. We will not repeat that discussion here, but focus on the new features that appear for B_N and D_N ensembles.

We discuss only potentials with the symmetry $W(x) = W(-x)$. In particular, $W(x)$ is a polynomial of even degree.

5.1 Saddle-point approximation

Since the degree d of $W(x)$ is even, we have an odd number $c = d - 1$ of critical points. In particular, due to the \mathbb{Z}_2 symmetry of the potential there is one critical point at $\text{Im}(x) = \text{Re}(x) = 0$. We denote the set of critical points as $\mu^{(k)}$ with $k \in \{-\frac{c-1}{2}, \dots, 0, \dots, \frac{c-1}{2}\}$, hence $\mu^{(-k)} = -\mu^{(k)}$. The saddle-point approximation requires us to distribute the N eigenvalues between the c critical points. Let us denote the number of eigenvalues located around $\mu^{(k)}$ as N_k . In contrast to the A_N case, we have to impose some additional constraints onto the eigenvalue distribution. This will allow the B_N and D_N ensembles to be dual to, both, $\mathcal{N} = 2$ SO/Sp gauge theories with adjoint broken to $\mathcal{N} = 1$ by a tree-level potential of the form $W(x)$ and to topological string orientifolds. From an orientifold point of view, it is more convenient to work in the quotient space perspective. In particular, for the eigenvalue ensembles this allows to avoid to deal with the interactions between “mirror” eigenvalues under the \mathbb{Z}_2 identification of cuts which the duality to orientifolds requires. Following [9], we implement the quotient into the eigenvalue ensemble by localizing the eigenvalues around the quotient set of critical points, *i.e.*, we take the eigenvalues to be localized around $(c+1)/2$ of the critical points such that

$$N = N_0 + \dots + N_{\frac{c-1}{2}}.$$

The A_N condition for consistency of the β -deformation of filling all cuts stated in the introduction then changes in the B_N/D_N case to filling only the quotient set of cuts.

The partition function can then be evaluated by considering small fluctuations $y_n^{(k)}$, with $k \in \{0, \dots, \frac{c-1}{2}\}$, around the critical points, *i.e.*, we set

$$(\lambda_1, \lambda_2, \dots, \lambda_N) = \left(\mu^{(0)} + y_1^{(0)}, \dots, \mu^{(0)} + y_{N_0}^{(0)}, \dots \right).$$

Under this decomposition we have that

$$\begin{aligned}\Delta_-(\lambda) &\rightarrow \prod_{k=0}^{\frac{c-1}{2}} \prod_{i < j}^{N_k} \left(y_i^{(k)} - y_j^{(k)} \right) \prod_{0 \leq m < n \leq \frac{c-1}{2}} \prod_{i=1}^{N_m} \prod_{j=1}^{N_n} \left(\mu^{(m)} - \mu^{(n)} + y_i^{(m)} - y_j^{(n)} \right), \\ \Delta_+(\lambda) &\rightarrow \prod_{k=0}^{\frac{c-1}{2}} \prod_{i < j}^{N_k} \left(2\mu^{(k)} + y_i^{(k)} + y_j^{(k)} \right) \prod_{0 \leq m < n \leq \frac{c-1}{2}} \prod_{i=1}^{N_m} \prod_{j=1}^{N_n} \left(\mu^{(m)} + \mu^{(n)} + y_i^{(m)} + y_j^{(n)} \right).\end{aligned}\tag{5.1}$$

Hence,

$$|P_{(b,d)}(\lambda)|^{2\beta} \rightarrow \left(\prod_{k=1}^{\frac{c-1}{2}} \Delta_-(y^{(k)}) \Delta_-(y^{(0)}) (\Delta_+(y^{(0)}))^{b+d} \prod_{i=1}^N (y_i^{(0)})^b \right)^{2\beta} \exp \mathcal{I}(y), \tag{5.2}$$

with interaction term

$$\begin{aligned}\mathcal{I}(y) &= -2\beta \sum_{0 \leq m < n \leq \frac{c-1}{2}} \sum_{l=1}^{\infty} \sum_{r=0}^l \frac{(-1)^r}{l(\mu^{(m)} - \mu^{(n)})^l} \binom{l}{r} S_r^{(m)} S_{l-r}^{(n)} \\ &\quad - \beta(b+d) \sum_{k=1}^{\frac{c-1}{2}} \sum_{l=1}^{\infty} \frac{(-1)^l}{2^l l (\mu^{(k)})^l} \sum_{r=0}^l \binom{l}{r} S_r^{(k)} S_{l-r}^{(k)} + \beta(d-b) \sum_{k=1}^{\frac{c-1}{2}} \sum_{l=1}^{\infty} \frac{(-1)^l}{l (\mu^{(k)})^l} S_l^{(k)} \\ &\quad - 2\beta(b+d) \sum_{0 \leq m < n \leq \frac{c-1}{2}} \sum_{l=1}^{\infty} \frac{(-1)^l}{l(\mu^{(m)} + \mu^{(n)})^l} \sum_{r=0}^l \binom{l}{r} S_r^{(m)} S_{l-r}^{(n)} + \text{const.}\end{aligned}\tag{5.3}$$

The potential decomposes as

$$\sum_{i=1}^N W(\lambda_i) \rightarrow \sum_{k=0}^{\frac{c-1}{2}} \sum_{n=1}^{\infty} \frac{(\partial^n W)(\mu^{(k)})}{n!} S_n^{(k)} + \text{const.}\tag{5.4}$$

Note that the B_N , respectively D_N ensemble decomposes in the saddle-point approximation into $(c-1)/2$ A_N and a single B_N , respectively D_N eigenvalue ensembles, which are coupled via the interaction term $\mathcal{I}(y)$, as expected. Hence, after an appropriate g_s dependent redefinition of $S_i^{(m)}$, and expansion in g_s to bring down powers of $S_i^{(m)}$, the partition function of the eigenvalue ensemble (2.4) reduces in the saddle-point approximation to a sum over Gaussian correlators which are determinable via the results of section 4. Due to the normalization of the Gaussian correlators, the resulting partition function has to be supplemented for each cut by a factor of $Z_{\mathcal{G}}(\beta)$ (defined in (2.1)) with \mathcal{G} either B_N or D_N for the fixed cut and else A_N .

5.2 Examples: B_N and D_N quartic

Let us now consider an example in more detail. The simplest non-trivial \mathbb{Z}_2 symmetric example is given by the quartic potential

$$W(\lambda) = \frac{\beta}{g_s} g \left(\frac{1}{4} \lambda^4 - \frac{\delta^2}{2} \lambda^2 \right). \quad (5.5)$$

Clearly, $W(-\lambda) = W(\lambda)$ and the set of critical points $\mu^{(k)}$ consists of

$$\mu^{(-1)} = -\delta, \quad \mu^{(0)} = 0, \quad \mu^{(1)} = \delta,$$

hence possesses a three-cut structure. The partition functions $Z_{(b,d)}$ for A_N , B_N and D_N measure can be obtained in a perturbative fashion as outlined in the previous section. While one has to fill in the A_N case all three cuts as discussed in the introduction, for B_N and D_N one has to fill only two of the cuts in order to incorporate the \mathbb{Z}_2 quotient as mentioned in section 5.1.

For the A_N case, let us just quote the relevant observation without giving any further details. Namely, we observe that if we fill all three-cuts, the disk sector (g_s^{-1}) of the corresponding free energy is a combination of closed periods, *i.e.*,

$$\tilde{\mathcal{F}}_A^{(1/2)} = \frac{1}{2} \left(1 - \frac{1}{\beta} \right) \left(\frac{\partial \mathcal{F}_A^{(0)}}{\partial \tilde{S}_{-1}} + \frac{\partial \mathcal{F}_A^{(0)}}{\partial \tilde{S}_0} + \frac{\partial \mathcal{F}_A^{(0)}}{\partial \tilde{S}_1} \right),$$

where as usual $\tilde{S}_i := N_i g_s$. This indicates that one has to perform the additional quantum shifts

$$S_i := \left(N_i - \frac{1}{2} \left(1 - \frac{1}{\beta} \right) \right) g_s, \quad (5.6)$$

in the large N limit. Indeed, after the shifts, one obtains an expansion into only even powers of g_s of the free energy \mathcal{F}_A , as is necessary for a well-defined BPS index interpretation of the corresponding partition function.

Let us now consider the B_N and D_N cases. For the quartic potential (5.5), the measure specializes to

$$[dy^{(0)}][dy^{(1)}] \left(\Delta_-(y^{(1)}) \Delta_-(y^{(0)}) (\Delta_+(y^{(0)}))^{b+d} \prod_{i=1}^N (y_i^{(0)})^b \right)^{2\beta},$$

the interaction term (5.3) reads

$$\begin{aligned} \mathcal{I}(y) = & -2\beta \sum_{l=1}^{\infty} \sum_{r=0}^l \frac{(-1)^l}{l\delta^l} \binom{l}{r} (1 + (b+d)(-1)^r) S_r^{(0)} S_{l-r}^{(1)} \\ & - \beta(b+d) \sum_{l=1}^{\infty} \frac{(-1)^l}{2^l l \delta^l} \sum_{r=0}^l \binom{l}{r} S_r^{(1)} S_{l-r}^{(1)} + \beta(d-b) \sum_{l=1}^{\infty} \frac{(-1)^l}{l\delta^l} S_l^{(1)} + \text{const.}, \end{aligned} \quad (5.7)$$

and the potential contribution (5.4) is given by

$$\mathcal{W}(y) = - \sum_{n=1}^{\infty} \frac{1}{n!} ((\partial^n W)(0) S_n^{(0)} + (\partial^n W)(\delta) S_n^{(1)}) .$$

After performing the rescalings

$$S_n^{(0)} \rightarrow \left(-\frac{g_s}{\beta g \delta^2} \right)^{n/2} S_n^{(0)}, \quad S_n^{(1)} \rightarrow \left(\frac{g_s}{2\beta g \delta^2} \right)^{n/2} S_n^{(1)},$$

the partition functions can be expanded in g_s and reduce to a sum over Gaussian correlators, which one can efficiently calculate following section 4. For the reader's convenience, the explicit free energies to some lower order in g_s are given in appendix A. Defining

$$S_0 = 2N_0 g_s, \quad S_1 = N_1 g_s, \quad (5.8)$$

we obtain

$$\tilde{\mathcal{F}}_B^{(0)}(S_0, S_1) = \tilde{\mathcal{F}}_D^{(0)}(S_0, S_1) = \frac{1}{2} \tilde{\mathcal{F}}_A^{(0)}(S_0, S_1, S_{-1} = S_1), \quad (5.9)$$

which is the expected tree-level result from a topological string orientifold perspective. The first order open string corrections (g_s^{-1}) read

$$\begin{aligned} \tilde{\mathcal{F}}_B^{(1/2)} &= \frac{1}{2} \left(2 - \frac{1}{\beta} \right) \frac{\partial \mathcal{F}_B^{(0)}}{\partial \tilde{S}_0} + \frac{1}{2} \left(1 - \frac{1}{\beta} \right) \frac{\partial \mathcal{F}_B^{(1)}}{\partial \tilde{S}_1}, \\ \tilde{\mathcal{F}}_D^{(1/2)} &= -\frac{1}{2\beta} \frac{\partial \mathcal{F}_D^{(0)}}{\partial \tilde{S}_0} + \frac{1}{2} \left(1 - \frac{1}{\beta} \right) \frac{\partial \mathcal{F}_D^{(1)}}{\partial \tilde{S}_1}, \end{aligned} \quad (5.10)$$

and are combinations of closed string periods. Note that for $\beta = 1$ we have that $\tilde{\mathcal{F}}_B^{(1/2)} = -\tilde{\mathcal{F}}_D^{(1/2)} = -\frac{1}{4} \partial_{S_0} \mathcal{F}_A(S_0, S_1, S_{-1} = S_1)$, confirming the earlier results of [8, 47, 48, 9]. In the dual topological string orientifold, the sign difference translates into the two possible choices of charge of the orientifold fixed-plane. Similar as for the A_N case, the

order g_s^{-1} given in (5.10) suggests to perform the additional quantum shifts as in (5.6) such that

$$\mathcal{F}_B^{(1/2)} = -\frac{1}{2} \frac{\partial \tilde{\mathcal{F}}_A^{(0)}(S_0, S_1, S_{-1} = S_1)}{\partial \tilde{S}_0}, \quad \tilde{\mathcal{F}}_D^{(1/2)} = \frac{1}{2} \frac{\partial \tilde{\mathcal{F}}_A^{(0)}(S_0, S_1, S_{-1} = S_1)}{\partial \tilde{S}_0}, \quad (5.11)$$

and the relation $\mathcal{F}_B^{(1/2)} = -\mathcal{F}_D^{(1/2)}$ continues to hold under the β -deformation. In particular, the open string contribution at order g_s^{-1} is independent of β . However, for higher orders in g_s , one has that generally

$$\mathcal{F}_B^{(g>1/2)}(\beta) \neq \mathcal{F}_D^{(g>1/2)}(\beta),$$

and equality (up to overall sign) only for $\beta = 1$. Since the open string contribution is trivial (*i.e.*, it is a closed period), it is more convenient to shift away the complete open string contribution such that $\mathcal{F}_B^{(g/2)} = \mathcal{F}_D^{(g/2)} = 0$, for g odd. Hence, in this specific gauge the free energies possess an expansion into even powers of g_s only. If necessary, the open string contribution can be easily reinstated via performing a reverse shift. The main advantage of this shift is that it allows us to utilize the usual holomorphic anomaly of [33] instead of the extended holomorphic of [37] to reproduce the B_N and D_N partition functions in the B-model. On a technical level, the former is easier to deal with. However, we like to stress that the latter is more general, since it is expected to capture the partition function independent of any shift of parameters, similar as observed at hand of gauge theory in [34, 35].

Finally, let us comment on the Nekrasov-Shatashvili limit of the free energies (in the gauge with an even power g_s expansion). In our parameterization the limit of [5] corresponds to

$$\mathcal{W}_G^{(g)} := \lim_{\beta \rightarrow 0} \beta^g \mathcal{F}_G^{(g)}(\beta). \quad (5.12)$$

From our explicit computations we observe that

$$\frac{1}{2} \mathcal{W}_A^{(g)} = \mathcal{W}_B^{(g)} = \mathcal{W}_D^{(g)}. \quad (5.13)$$

A similar non-uniqueness property of the limit has been already observed at hand of gauge theory on ALE space in [42], and is in fact as expected. This is because, since $g_s \rightarrow 0$ (in order to keep $\hbar := \frac{g_s}{\sqrt{\beta}}$ fixed, *cf.*, (1.1)), the Nekrasov-Shatashvili limit, and hence $\mathcal{W}^{(g)}$, is intrinsically of tree-level nature. More specifically, the limit corresponds to a (semi-classical limit of a) quantization of the spectral curve of the respective eigenvalue ensemble, following [3]. Since the spectral curves of the A_N , B_N and D_N

ensembles (under a proper \mathbb{Z}_2 identification of the A_N spectral curve) are identical (*cf.*, (5.9)), so should be the quantization thereof. The relation (5.13) shows that this is indeed the case.

6 B-model verification of B_N and D_N quartic

6.1 Tree-level geometry

The dual tree-level geometry of the quartic eigenvalue ensemble with B_N and D_N measure with $\beta = 1$ has been discussed already in the literature to some extent (*cf.*, [49, 8, 9]). Since the power of β is only relevant at one-loop and beyond, we essentially can borrow the known tree-level results.

The periods of the dual geometry of the eigenvalue ensemble with potential $W(x)$ of the form (5.5) are given in terms of the periods of the hyperelliptic curve

$$y = M(x)\sqrt{\sigma(x)}, \quad (6.1)$$

with

$$M(x) = g, \quad \sigma(x) = \frac{1}{g^2} (W'(x)^2 + f(x)), \quad (6.2)$$

and where $f(x)$ is a degree two polynomial. In particular, the moment function $M(x)$ is a constant because we fill all cuts. For the quartic, the curve (6.1) is of genus two. The effective one-form of the dual geometry reads

$$\omega = y dx, \quad (6.3)$$

which we express in terms of the six branch points x_i of the curve as

$$\omega = g \sqrt{\prod_{i=1}^6 (x - x_i)} dx.$$

The cuts are chosen to be $[x_1, x_2]$, $[x_3, x_4]$ and $[x_5, x_6]$ on the real axis. Imposing the \mathbb{Z}_2 symmetry under $x \rightarrow -x$ of the x -plane (this requires that $f(x)$ is even, *i.e.*, $f(x) = b_2 x^2 + b_0$ with b_i parameterizing the complex structure) leads to the identification $x_1 \leftrightarrow -x_6$, $x_2 \leftrightarrow -x_5$ and $x_3 \leftrightarrow -x_4$, yielding the one-form

$$\omega = g \sqrt{(x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)} dx.$$

The \mathbb{Z}_2 symmetric x -plane of the geometry is illustrated in figure 1.

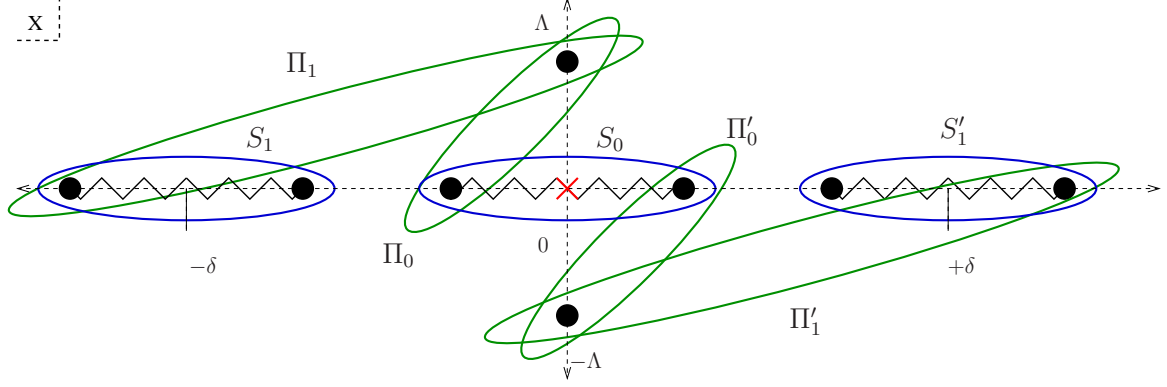


Figure 1: The symmetric x -plane of the quartic with cuts, period contours and reflection symmetry indicated (covering space perspective).

One should note that if one adjusts such that $b_2 = 0$, the curve (6.1) takes the form of the Seiberg-Witten curve of four dimensional $\mathcal{N} = 2$ $SU(3)$ gauge theory, with choice of Coulomb parameters $a_1 = -a_2$ and $a_3 = 0$ (the zeroes of $W'(x)$ correspond to the a_i), which can also be matched to the curve of $Sp(2)$. It would be interesting to extract the gauge theory gauge coupling and 1-loop gravitational correction following [50, 20] and see if one can match to a Ω -deformed gauge theory, as is the case for the cubic and Ω -deformed $SU(2)$ [3]. However, since $a_3 = 0$, we are not on the Coloumb branch, and things are expected to be a bit more tricky. Therefore we will not follow this rather interesting direction further in this work.

Comparison with the one-form (6.3) expressed via (6.2) gives the relation

$$\delta^2 = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) . \quad (6.4)$$

In order to explicitly calculate the period integrals, it is useful to change variables to

$$z_0 = x_3, \quad z_1 = \frac{1}{2}(x_2 - x_1), \quad \mathcal{I} = \frac{1}{2}(x_1 + x_2) . \quad (6.5)$$

Hence, we set

$$x_3 = z_0, \quad x_1 = \mathcal{I} - z_1, \quad x_2 = \mathcal{I} + z_1 .$$

Using (6.4), we can infer for \mathcal{I} the relation

$$\mathcal{I} = \pm \sqrt{\delta^2 - \frac{1}{2}z_0^2 - z_1^2} .$$

The discriminant Δ of the algebraic curve (6.1) reads in the z_i coordinates (6.5)

$$\Delta = 4z_0^2 z_1^4 \Delta_1^2 \Delta_2^2 \Delta_3^4, \quad (6.6)$$

with components

$$\begin{aligned} \Delta_1 &= z_0^2 + 2z_1^2 - 2\delta^2, & \Delta_2 &= z_0^2 + 4z_1^2 - 2\delta^2, \\ \Delta_3 &= 9z_0^4 + 4z_0^2(2z_1^2 - 3\delta^2) + 4(-2z_1^2 + \delta^2)^2. \end{aligned} \quad (6.7)$$

In particular, we have that $\Delta_1 = -2\mathcal{I}^2$.

The A-periods of the curve (6.1) with one-form (6.3) are taken to be

$$S_0 = \frac{1}{2\pi i} \int_{-x_3}^{x_3} \omega, \quad S_1 = \frac{1}{2\pi i} \int_{x_2}^{x_1} \omega. \quad (6.8)$$

It is not hard to explicitly evaluate the integrals in the coordinates (6.5) for small z_i . The first few terms read

$$\begin{aligned} S_0(z_i) &= -\frac{g\delta^2}{4} z_0^2 + \frac{g}{2} z_0^2 z_1^2 + \frac{3g}{16} z_0^4 + \frac{g}{8\delta^2} z_0^4 z_1^2 + \frac{g}{4\delta^4} z_0^4 z_1^4 + \frac{g}{8\delta^4} z_0^6 z_1^2 + \dots, \\ S_1(z_i) &= \frac{g\delta^2}{2} z_1^2 - \frac{g}{2} z_0^2 z_1^2 - \frac{g}{2} z_1^4 - \frac{g}{16\delta^2} z_0^4 z_1^2 - \frac{g}{8\delta^4} z_0^4 z_1^4 - \frac{g}{16\delta^4} z_0^6 z_1^2 + \dots, \end{aligned} \quad (6.9)$$

Note that the two periods are related via the identity

$$S_1 + \frac{1}{2}S_0 = \frac{g}{32}(z_0^2 - 4z_1^2)(3z_0^2 + 4z_1^2 - 4\delta^2). \quad (6.10)$$

Inversion of (6.9) yields the so-called mirror maps $z_i(S_i)$.

Similarly, it is not hard to explicitly evaluate the B-periods

$$\Pi_0 = \frac{1}{2} \int_{x_3}^{\Lambda} \omega, \quad \Pi_1 = \int_{x_1}^{\Lambda} \omega,$$

where $\Lambda \rightarrow \infty$ is a cutoff. Note the additional factor of $1/2$ we introduced for Π_0 . Its origin can be most easily seen at hand of figure 1. The period Π_0 (without the $1/2$) in the quotient space corresponds in the covering space actually to $2\Pi_0$ and not just Π_0 .

We obtain in terms of the flat coordinates S_i for the B-periods

$$\begin{aligned} \Pi_0(S_i) &= \frac{g}{8}(\Lambda^2 - 2\delta^2)\Lambda^2 - \frac{1}{2}S_0 \log g - 2(S_1 + \frac{1}{2}S_0) \log \Lambda \\ &\quad + 2(S_1 - \frac{1}{2}S_0) \log \delta + P_0(S_0, S_1), \\ \Pi_1(S_i) &= \frac{g}{4}(\delta^2 - \Lambda^2)^2 - 4(S_1 + \frac{1}{2}S_0) \log \Lambda + 2(S_1 + S_0) \log \delta + S_1 \log \left(\frac{2}{g\delta^2} \right) \\ &\quad + P_1(S_0, S_1), \end{aligned} \quad (6.11)$$

with $P_i(S_0, S_1) = S_i \log S_i + \sum_{n,m \geq 0} \frac{c_i(n,m)}{(g\delta^2)^{n+m-1}} S_0^n S_1^m$ and $c_i(n, m)$ constants .

Invoking the usual special geometry relation

$$\partial_{S_i} \mathcal{F}^{(0)} = \Pi_i(S_0, S_1) ,$$

the prepotential $\mathcal{F}^{(0)}$ can be determined, and indeed matches the results of section 5.2. Note that the $P_i(S_0, S_1)$ can be expressed in terms of the flat-coordinate Yukawa couplings $C_{S_i S_j S_k}$ as $P_i(S_0, S_1) = \int dS_i dS_j dS_k C_{S_i S_j S_k}$. Closed expressions for the Yukawa couplings in z_i coordinates, *i.e.*, $C_{z_i z_j z_k} := D_{z_i} D_{z_j} D_{z_k} \mathcal{F}^{(0)}(z_i)$ with D_{z_i} denoting the covariant derivative (*cf.*, [33]), can be found to be

$$\begin{aligned} C_{z_0 z_0 z_0} &= -\frac{z_0(9z_0^6 + 6z_0^4(-5 + z_1^2) + 8(1 - 2z_1^2)^2(-1 + z_1^2) + 4z_0^2(7 - 8z_1^2 + 4z_1^4))}{32\mathcal{I}^2} , \\ C_{z_1 z_1 z_1} &= -\frac{z_1(3z_0^6 + 8(-1 + 2z_1^2)^3 + 2z_0^4(-7 + 10z_1^2) + 4z_0^2(5 - 16z_1^2 + 12z_1^4))}{4\mathcal{I}^2} , \\ C_{z_0 z_0 z_1} &= \frac{z_1 z_0^2(-2 + 3z_0^2)(-2 + z_0^2 + 4z_1^2)}{8\mathcal{I}^2} , \\ C_{z_0 z_1 z_1} &= \frac{z_0 z_1^2(z_0^4 - 4(1 - 2z_1^2)^2)}{4\mathcal{I}^2} . \end{aligned} \tag{6.12}$$

The remaining couplings follow by symmetry. We have also set for simplicity $g = \delta = 1$.

6.2 One-loop

Having the tree-level data at hand, it is straight-forward to evaluate the solution to the 1-loop holomorphic anomaly equation of [51]

$$\mathcal{F}^{(1)}(z; \beta) = \frac{1}{2} \log \det G + a^{(1)}(z; \beta) , \tag{6.13}$$

with $G_{ij} := \partial_{S_i} z_j$ and $a^{(1)}(z; \beta)$ denoting the 1-loop holomorphic ambiguity. The ambiguity can be parameterized in terms of the discriminant loci Δ_i given in (6.7) as

$$a^{(1)}(z; \beta) = \nu_0 \log z_0 + \nu_1 \log z_1 + \kappa_1 \log \Delta_1 + \kappa_2 \log \Delta_2 + \kappa_3 \log \Delta_3 .$$

From the eigenvalue ensemble results of section 5.2 we deduce that under fixing parameters ν_i and κ_i in the B_N case to

$$\begin{aligned} \nu_0 &= \frac{1}{24} \left(\frac{1}{\beta} + 4\beta \right) - \frac{1}{2} , \quad \nu_1 = \frac{1}{12} \left(\frac{1}{\beta} + \beta \right) - \frac{1}{2} , \\ \kappa_1 &= \nu_0 , \quad \kappa_2 = \frac{1}{2} \nu_1 + \frac{1}{4} , \quad \kappa_3 = \nu_1 , \end{aligned} \tag{6.14}$$

and for D_N to

$$\begin{aligned}\nu_0 &= \frac{1}{24} \left(\frac{1}{\beta} - 8\beta \right), \quad \nu_1 = \frac{1}{12} \left(\frac{1}{\beta} + \beta \right) - \frac{1}{2}, \\ \kappa_1 &= \nu_0, \quad \kappa_2 = \frac{1}{2}\nu_1 + \frac{1}{4}, \quad \kappa_3 = \nu_1,\end{aligned}\tag{6.15}$$

the previous results can be reproduced. We observe that the ambiguities for both cases differ in general only in ν_0 . Further, note that only in the special case of $\beta = 1$ and in the Nekrasov-Shatashvili limit ($\beta \rightarrow 0$) the ν_0 of both cases are equal and the relation

$$\mathcal{F}_B^{(1)} = \mathcal{F}_D^{(1)},$$

holds, as expected from the relations (3.19) and (5.13). For general β this equality will not hold anymore. Also note that in the Nekrasov-Shatashvili limit (5.12) the 1-loop amplitude (6.13) becomes purely holomorphic, *i.e.*,

$$\mathcal{W}^{(1)}(z) = \frac{1}{24} \log z_0 + \frac{1}{12} \log z_1 + \frac{1}{24} \log (\Delta_1 \Delta_2) + \frac{1}{12} \log \Delta_3.$$

The reason being that in this limit the 1-loop anomaly equation reduces to

$$\bar{\partial}_i \partial_j \mathcal{W}^{(1)}(z, \bar{z}) = 0.$$

To conclude this section, it is interesting to compare the ν_0 coefficients we found to the corresponding $\Psi_{\mathcal{G}}^{(0)}$ coefficients of section 3, given in (3.12) and (3.20). Up to an addition of $1/2$ they match. We attribute the additional $1/2$ to an artifact of our expansion at the 1-loop level and/or to the chosen parameterization. Mainly because the same mismatch by $1/2$ occurs for ν_1 , which should be equal to $\Psi_A^{(0)}$.

7 Conclusion

In this work we initiated the study of β -ensembles with B_N and D_N measure beyond tree-level. For that purpose, we generalized the calculation of the β -deformed A_N partition function of [18], which makes use of a saddle-point approximation and Ward identities, to the B_N and D_N cases. At hand of the quartic, we found that the resulting free energies possess an expansion into even powers of g_s only, under a specific choice of 't Hooft parameters. This is as expected, since the g_s^{-1} sector is a closed period and should be removable via an appropriate shift of parameters following [35]. The absence of an odd sector in g_s allowed us to invoke the usual holomorphic anomaly equation to

reproduce the 1-loop sector (g_s^0) of the quartic, albeit with new boundary conditions (holomorphic ambiguity) which have not appeared (to our knowledge) before. The boundary conditions are related to a large N expansion of the Macdonald integral. We expect that the higher genus coefficients of the Macdonald integral expansions will provide boundary conditions for the higher loop amplitudes expanded near some of the other points in moduli space. However, so far we have not pushed the holomorphic anomaly calculation for the quartic beyond genus one, and it would be interesting to do so. Our results indicate that the β -deformation of the B_N and D_N ensembles, which for $\beta = 1$ correspond to topological string orientifolds, is consistent.

The Gaussian integrals also allowed us to extrapolate the B_N and D_N ensembles to toric settings. We found that the D_N case (under an appropriate shift) should correspond to the topological string with boundary conditions at the conifold point provided by the orbifold branch of the $c = 1$ string. The B_N case corresponds to the usual circle branch, similar to the standard refined topological string related to A_N ensemble. In the toric setting, our results indicate that the equivalence between B_N/D_N and topological string orientifolds is not general. Rather the $c = 1$ moduli space appears to yield an independent deformation space of topological string theories, with only accidental correspondence to topological string orientifolds for specific simple geometries.

It seems likely that one may also explore ensembles with E_6 , E_7 and E_8 type measure in a similar fashion. Presumably, these ensembles are related to the three discrete points in $c = 1$ moduli space (and should not possess a consistent β -deformation).

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A Free energies

The explicit results for the perturbative expansion of the free energies of the B_N and D_N eigenvalue ensemble with quartic potential discussed in section 5.2 are given below. Note that the expressions given do not take the normalization of (4.1) into account. The additional contribution from the normalization to the shifted free energy is given in terms of an asymptotic expansion of the respective Macdonald integral given in (2.2).

$$\begin{aligned}
\tilde{\mathcal{F}}_B(N_i, \beta) = & -\frac{g_s}{4\delta^4\beta} \left(8N_0^3\beta^2 - 2N_0^2\beta(2\beta(8N_1+1)-5) + N_0 \left(8\beta^2(N_1-3)N_1 + \beta(8N_1-2)+3 \right) + (2\beta-1)N_1(2\beta(N_1-2)+3) \right) \\
& + \frac{g_s^2}{4g^2\delta^8\beta^2} \left(36N_0^4\beta^3 - 4N_0^3\beta^2(2\beta(28N_1+5)-19) + N_0^2\beta \left(12\beta^2(12N_1^2-16N_1+1) + \beta(24N_1-50)+53 \right) \right. \\
& + N_0 \left(-8\beta^3N_1(2N_1^2-15N_1+19) + \beta^2(-48N_1^2+200N_1+6) - \beta(88N_1+15)+12 \right) - (2\beta-1)N_1 \left(4\beta^2(N_1^2-5N_1+5) \right. \\
& \left. \left. + \beta(15N_1-29)+12) \right) \right) \\
& - \frac{g_s^3}{12g^3\delta^{12}\beta^3} \left(864N_0^5\beta^4 - 8N_0^4\beta^3(4\beta(233N_1+49)-331) + 8N_0^3\beta^2 \left(8\beta^2(131N_1^2-102N_1+16) - 4\beta(29N_1+103)+383 \right) \right. \\
& - 4N_0^2\beta \left(4\beta^2(165N_1^2-888N_1-80) + 4\beta^3(152N_1^3-558N_1^2+640N_1+15) + 2\beta(965N_1+286)-393 \right) + N_0 \left(8\beta^2(419N_1^2 \right. \\
& - 798N_1+48) + 32\beta^4N_1(5N_1^3-66N_1^2+206N_1-165) + 8\beta^3(112N_1^3-1050N_1^2+1208N_1-15) + 12\beta(88N_1-43)+297 \right) \\
& \left. + (2\beta-1)B \left(4\beta^2(66N_1^2-314N_1+307) + 8\beta^3(5N_1^3-44N_1^2+108N_1-74) + 6\beta(86N_1-163)+297 \right) \right) \\
& - \frac{g_s^4}{8g^4\delta^{16}\beta^4} \left(-6048N_0^6\beta^5 + 16N_0^5\beta^4(\beta(4232N_1+982)-1549) - 4N_0^4\beta^3 \left(4\beta^2(7088N_1^2-3492N_1+1033) - 2\beta(3596N_1 \right. \right. \\
& + 5973) + 10325) + 8N_0^3\beta^2 \left(\beta^2(3232N_1^2-29640N_1-4326) + 16\beta^3(456N_1^3-1088N_1^2+1275N_1+65) + \beta(18132N_1 \right. \\
& + 6853) - 4350) - N_0^2\beta \left(4\beta^2(27696N_1^2-29580N_1+5981) + 32\beta^3(740N_1^3-7254N_1^2+6111N_1-325) + 16\beta^4(628N_1^4 \right. \\
& - 4216N_1^3+11172N_1^2-7232N_1+105) + 48\beta(131N_1-581)+14691) + N_0 \left(48\beta^2(515N_1^2-1808N_1-112) \right. \\
& + 16\beta^3(1474N_1^3-7593N_1^2+9782N_1+195) + 32\beta^5N_1(14N_1^4-279N_1^3+1522N_1^2-2956N_1+1769) + 8\beta^4(488N_1^4 \\
& - 7680N_1^3+22104N_1^2-17872N_1-105) + 3\beta(7520N_1+1743)-2448) + (2\beta-1)N_1 \left(6\beta^2(632N_1^2-2905N_1+2799) \right. \\
& \left. + 4\beta^3(279N_1^3-2318N_1^2+5492N_1-3695) + 16\beta^4(7N_1^4-93N_1^3+398N_1^2-658N_1+353) + \beta(5229N_1-9795)+2448 \right) \right) \\
& - \frac{g_s^5}{10g^5\delta^{20}\beta^5} \left(93312N_0^7\beta^6 - 288N_0^6\beta^5(2\beta(2252N_1+559)-1685) + 16N_0^5\beta^4 \left(24\beta^2(7711N_1^2-2423N_1+1250) \right. \right. \\
& - 2\beta(31728N_1+40813)+67259) - 40N_0^4\beta^3 \left(4\beta^2(1605N_1^2-47267N_1-9059) + 12\beta^3(4784N_1^3-8246N_1^2+10442N_1 \right. \\
& + 803) + 2\beta(62029N_1+26899)-32463) + 4N_0^3\beta^2 \left(40\beta^2(35489N_1^2-20156N_1+10312) + 40\beta^3(5544N_1^3-66014N_1^2 \right. \\
& + 39818N_1-5027) + 48\beta^4(3670N_1^4-16580N_1^3+41550N_1^2-21780N_1+863) - 30\beta(4710N_1+14933)+223489) \\
& - 2N_0^2\beta \left(60\beta^2(10356N_1^2-51699N_1-6938) + 20\beta^3(54900N_1^3-186600N_1^2+252646N_1+13849) + 40\beta^4(4038N_1^4 \right. \\
& - 61004N_1^3+137454N_1^2-109848N_1-2589) + 16\beta^5(2568N_1^5-26610N_1^4+120040N_1^3-192810N_1^2+109832N_1+945) + \\
& \beta(982802N_1+373894)-164865) + N_0 \left(4\beta^2(246049N_1^2-415992N_1+38745) + 120\beta^3(4782N_1^3-34959N_1^2+35464N_1 \right. \\
& - 1034) + 8\beta^4(35850N_1^4-334320N_1^3+989930N_1^2-715820N_1+7767) + 64\beta^6N_1(42N_1^5-1158N_1^4+9500N_1^3-31960N_1^2 \\
& + 45679N_1-22355) + 16\beta^5(2064N_1^5-47710N_1^4+240920N_1^3-460650N_1^2+265116N_1-945) + 30\beta(6880N_1-4111) \\
& + 50139) + (2\beta-1)N_1 \left(4\beta^2(27945N_1^2-125282N_1+119563) + 8\beta^3(5895N_1^3-46950N_1^2+108328N_1-71927) \right. \\
& + 8\beta^4(1158N_1^4-14460N_1^3+59320N_1^2-95329N_1+50293) + 32\beta^5(21N_1^5-386N_1^4+2480N_1^3-7080N_1^2+9025N_1 \\
& - 4081) + 6\beta(20555N_1-38242)+50139) \right) \\
& + \mathcal{O}(g_s^6),
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_D(N_i, \beta) = & \frac{g_s}{4g\delta^4\beta} \left(-9N_0^3\beta^2 + N_1(3+2(-1+N_1)\beta) + 2N_0^2\beta(-5+8(1+2N_1)\beta) - N_0(3+2(-5+4N_1)\beta) + 8(1+N_1+N_1^2)\beta^2 \right) \\
& + \frac{g_s^2}{4g^2\delta^8\beta^2} \left(36N_0^4\beta^3 - 4N_0^3\beta^2(-19+28(1+2N_1)\beta) + N_1(12+15(-1+N_1)\beta) + 2(3-5N_1+2N_1^2)\beta^2 \right) \\
& + N_0^2\beta(53+2(-77+12N_1)\beta) + 4(29+36N_1+36N_1^2)\beta^2 - N_0(-12+(53+88N_1)\beta) + 2(-39-68N_1+24N_1^2)\beta^2 \\
& + 8(5+11N_1+3N_1^2+2N_1^3)\beta^3 \\
& + \frac{g_s^3}{12g^3\delta^{12}\beta^3} \left(-864N_0^5\beta^4 + 8N_0^4\beta^3(466\beta(2N_1+1)-331) - 8N_0^3\beta^2 \left(4\beta^2(262N_1^2+262N_1+189) - 4\beta(29N_1+260) + 383 \right) \right. \\
& + 4N_0^2\beta \left(20\beta^2(33N_1^2-174N_1-109) + 4\beta^3(152N_1^3+228N_1^2+622N_1+273) + 2\beta(965N_1+781) - 393 \right) \\
& - N_0 \left(8\beta^2(419N_1^2+98N_1+398) + 8\beta^3(112N_1^3-546N_1^2-412N_1-381) + 32\beta^4(5N_1^4+10N_1^3+80N_1^2+75N_1+37) \right. \\
& \left. + 12\beta(88N_1-131) + 297) + N_1 \left(24\beta^2(11N_1^2-27N_1+16) + 8\beta^3(5N_1^3-22N_1^2+32N_1-15) + 516\beta(N_1-1) + 297 \right) \right) \\
& + \frac{g_s^4}{8g^4\delta^{16}\beta^4} \left(6048N_0^6\beta^5 - 16N_0^5\beta^4(2116\beta(2N_1+1)-1549) + 4N_0^4\beta^3 \left(4\beta^2(7088N_1^2+7088N_1+4771) - 2\beta(3596N_1+13449) \right. \right. \\
& + 10325) - 8N_0^3\beta^2 \left(\beta^2(3232N_1^2-34536N_1-21990) + 24\beta^3(304N_1^3+456N_1^2+1054N_1+451) + 3\beta(6044N_1+5433) \right. \\
& - 4350) + N_0^2\beta \left(12\beta^2(9232N_1^2+7404N_1+11455) + 32\beta^3(740N_1^3-5286N_1^2-5271N_1-4010) + 16\beta^4(628N_1^4+1256N_1^3 \right. \\
& + 7236N_1^2+6608N_1+3085) + 48\beta(131N_1-1483) + 14691) - N_0 \left(48\beta^2(515N_1^2-1296N_1-758) + 16\beta^3(1474N_1^3 \right. \\
& - 789N_1^2+5814N_1+3023) + 8\beta^4(488N_1^4-3552N_1^3-2496N_1^2-9680N_1-4401) + 32\beta^5(14N_1^5+35N_1^4+490N_1^3 \\
& + 700N_1^2+937N_1+353) + 3\beta(7520N_1+4897) - 2448) + N_1 \left(48\beta^2(79N_1^2-191N_1+112) + 12\beta^3(93N_1^3-398N_1^2 \right. \\
& + 565N_1-260) + 8\beta^4(14N_1^4-93N_1^3+234N_1^2-260N_1+105) + 5229\beta(N_1-1) + 2448) \left. \right) \\
& + \frac{g_s^5}{10g^5\delta^{20}\beta^5} \left(-93312N_0^7\beta^6 + 288N_0^6\beta^5(2252\beta(2N_1+1)-1685) - 16N_0^5\beta^4 \left(8\beta^2(23133N_1^2+23133N_1+14833) \right. \right. \\
& - 2\beta(31728N_1+85145) + 67259) + 40N_0^4\beta^3 \left(4\beta^2(1605N_1^2-60559N_1-38558) + 4\beta^3(14352N_1^3+21528N_1^2+ \right. \\
& 44618N_1+18721) + 2\beta(62029N_1+58538) - 32463) - 4N_0^3\beta^2 \left(20\beta^2(70978N_1^2+79294N_1+96033) + 80\beta^3(2772N_1^3 \right. \\
& - 29003N_1^2-31778N_1-21998) + 32\beta^4(5505N_1^4+11010N_1^3+52315N_1^2+46810N_1+20967) - 30\beta(4710N_1+34231) \\
& + 223489) + 2N_0^2\beta \left(60\beta^2(10356N_1^2-51393N_1-36176) + 20\beta^3(54900N_1^3+22896N_1^2+252052N_1+140723) \right. \\
& + 40\beta^4(4038N_1^4-38428N_1^3-45138N_1^2-110094N_1-50559) + 16\beta^5(2568N_1^5+6420N_1^4+63600N_1^3+88980N_1^2 \\
& + 110042N_1+40405) + \beta(982802N_1+915556) - 164865) - N_0 \left(\beta^2(984196N_1^2+99616N_1+937156) + 120\beta^3(4782N_1^3 \right. \\
& - 19317N_1^2-10208N_1-12766) + 8\beta^4(35850N_1^4-57660N_1^3+373880N_1^2+274580N_1+194147) + 16\beta^5(2064N_1^5 \\
& - 20630N_1^4-9820N_1^3-140460N_1^2-115344N_1-58455) + 64\beta^6(42N_1^6+126N_1^5+2730N_1^4+5250N_1^3+13309N_1^2 \\
& + 10705N_1+4081) + 30\beta(6880N_1-10991) + 50139) + N_1 \left(540\beta^2(207N_1^2-494N_1+287) + 120\beta^3(393N_1^3-1643N_1^2 \right. \\
& + 2284N_1-1034) + 24\beta^4(386N_1^4-2480N_1^3+6050N_1^2-6545N_1+2589) + 16\beta^5(42N_1^5-386N_1^4+1450N_1^3-2750N_1^2 \\
& + 2589N_1-945) + 123330\beta(N_1-1) + 50139) \left. \right) \\
& + \mathcal{O}(g_s^6).
\end{aligned}$$

References

- [1] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” Nucl. Phys. B **644** (2002) 3 [arXiv:hep-th/0206255].
- [2] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems,” arXiv:0909.2453 [hep-th].
- [3] M. Aganagic, M. C. N. Cheng, R. Dijkgraaf, D. Krefl and C. Vafa, “Quantum Geometry of Refined Topological Strings,” arXiv:1105.0630 [hep-th].
- [4] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. **91** (2010) 167 [arXiv:0906.3219 [hep-th]].
- [5] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].
- [6] M. Aganagic and S. Shakirov, “Knot Homology from Refined Chern-Simons Theory,” arXiv:1105.5117 [hep-th].
- [7] D. Ghoshal and C. Vafa, “ $C = 1$ string as the topological theory of the conifold,” Nucl. Phys. B **453**, 121 (1995) [arXiv:hep-th/9506122].
- [8] S. K. Ashok, R. Corrado, N. Halmagyi, K. D. Kennaway and C. Romelsberger, “Unoriented strings, loop equations, and $N = 1$ superpotentials from matrix models,” Phys. Rev. D **67** (2003) 086004 [arXiv:hep-th/0211291].
- [9] K. A. Intriligator, P. Kraus, A. V. Ryzhov, M. Shigemori and C. Vafa, “On low rank classical groups in string theory, gauge theory and matrix models,” Nucl. Phys. B **682** (2004) 45 [arXiv:hep-th/0311181].
- [10] N. Nekrasov and S. Shadchin, “ABCD of instantons,” Commun. Math. Phys. **252** (2004) 359 [arXiv:hep-th/0404225].
- [11] M. Aganagic and K. Schaeffer, “Orientifolds and the Refined Topological String,” arXiv:1202.4456 [hep-th].
- [12] K. Landsteiner, C. I. Lazaroiu and R. Tatar, “(Anti)symmetric matter and superpotentials from IIB orientifolds,” JHEP **0311** (2003) 044 [arXiv:hep-th/0306236].
- [13] K. Landsteiner and C. I. Lazaroiu, “On $Sp(0)$ factors and orientifolds,” Phys. Lett. B **588** (2004) 210 [arXiv:hep-th/0310111].
- [14] A. Brini, M. Marino and S. Stevan, “The Uses of the refined matrix model recursion,” arXiv:1010.1210 [hep-th].
- [15] V. Bouchard, A. Klemm, M. Marino and S. Pasquetti, “Remodeling the B-model,” Commun. Math. Phys. **287** (2009) 117 [arXiv:0709.1453 [hep-th]].
- [16] M. Marino, “Open string amplitudes and large order behavior in topological string theory,” JHEP **0803** (2008) 060 [arXiv:hep-th/0612127].

- [17] A. S. Alexandrov, A. Mironov and A. Morozov, “Partition functions of matrix models as the first special functions of string theory. 1. Finite size Hermitean one matrix model,” *Int. J. Mod. Phys. A* **19** (2004) 4127 [*Teor. Mat. Fiz.* **142** (2005) 419] [arXiv:hep-th/0310113].
- [18] A. Morozov and S. Shakirov, “The matrix model version of AGT conjecture and CIV-DV prepotential,” *JHEP* **1008** (2010) 066 [arXiv:1004.2917 [hep-th]].
- [19] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “Matrix model as a mirror of Chern-Simons theory,” *JHEP* **0402** (2004) 010 [arXiv:hep-th/0211098].
- [20] A. Klemm, M. Marino and S. Theisen, “Gravitational corrections in supersymmetric gauge theory and matrix models,” *JHEP* **0303** (2003) 051 [arXiv:hep-th/0211216].
- [21] I. G. Macdonald, “Some Conjectures for Root Systems,” *SIAM J. Math. Anal.*, Vol. 13, No. 6, November 1982
- [22] E. M. Opdam, “Some applications of hypergeometric shift operators,” *Invent. Math.* 98, 275-282, 1989
- [23] E. M. Opdam, “Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group,” *Compositio Math.* 85, 333-373, 1993
- [24] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, “Topological strings and integrable hierarchies,” *Commun. Math. Phys.* **261** (2006) 451 [arXiv:hep-th/0312085].
- [25] J. Harer and D. Zagier, “The Euler Characteristic of the Moduli Space of Curves,” *Inv. Math.* 85 (1986) 457-485
- [26] L. Chekhov and A. Zabrodin, “A Critical matrix model for nonoriented string,” *Mod. Phys. Lett. A* **6** (1991) 3143.
- [27] I. P. Goulden, J. L. Harer and D. M. Jackson, “A geometric parametrization for the virtual Euler characteristic of the moduli space of real and complex algebraic curves,” *Trans. Am. Math. Soc.* 353, 4405 (2001)
- [28] H. Ooguri and C. Vafa, “World sheet derivation of a large N duality,” *Nucl. Phys. B* **641** (2002) 3 [arXiv:hep-th/0205297].
- [29] D. Krefl, S. Pasquetti and J. Walcher, “The Real Topological Vertex at Work,” *Nucl. Phys. B* **833** (2010) 153 [arXiv:0909.1324 [hep-th]].
- [30] D. Krefl, “Wall Crossing Phenomenology of Orientifolds,” arXiv:1001.5031 [hep-th].
- [31] D. J. Gross and I. R. Klebanov, “One-dimensional String Theory On A Circle,” *Nucl. Phys. B* **344** (1990) 475.
- [32] M. -x. Huang, A. Klemm and S. Quackenbush, “Topological string theory on compact Calabi-Yau: Modularity and boundary conditions,” *Lect. Notes Phys.* **757** (2009) 45 [arXiv:hep-th/0612125].
- [33] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun. Math. Phys.* **165** (1994) 311 [arXiv:hep-th/9309140].

- [34] D. Krefl and J. Walcher, “Extended Holomorphic Anomaly in Gauge Theory,” *Lett. Math. Phys.* **95** (2011) 67 [arXiv:1007.0263 [hep-th]].
- [35] D. Krefl and J. Walcher, “Shift versus Extension in Refined Partition Functions,” arXiv:1010.2635 [hep-th].
- [36] M. -x. Huang and A. Klemm, “Direct integration for general Ω backgrounds,” arXiv:1009.1126 [hep-th].
- [37] J. Walcher, “Extended holomorphic anomaly and loop amplitudes in open topological string,” *Nucl. Phys. B* **817** (2009) 167 [arXiv:0705.4098 [hep-th]].
- [38] D. Krefl and J. Walcher, “The Real Topological String on a local Calabi-Yau,” arXiv:0902.0616 [hep-th].
- [39] P. H. Ginsparg, “Curiosities at $c = 1$,” *Nucl. Phys. B* **295** (1988) 153.
- [40] R. Gopakumar and C. Vafa, “Topological gravity as large N topological gauge theory,” *Adv. Theor. Math. Phys.* **2** (1998) 413 [arXiv:hep-th/9802016].
- [41] D. Ghoshal, D. P. Jatkar and S. Mukhi, “Kleinian singularities and the ground ring of $C=1$ string theory,” *Nucl. Phys. B* **395** (1993) 144 [arXiv:hep-th/9206080].
- [42] D. Krefl and S. -Y. D. Shih, “Holomorphic Anomaly in Gauge Theory on ALE space,” arXiv:1112.2718 [hep-th].
- [43] B. S. Acharya, M. Aganagic, K. Hori and C. Vafa, “Orientifolds, mirror symmetry and superpotentials,” arXiv:hep-th/0202208.
- [44] J. Walcher, “Evidence for Tadpole Cancellation in the Topological String,” arXiv:0712.2775 [hep-th].
- [45] A. Mironov, A. Morozov, A. Popolitov and S. .Shakirov, “Resolvents and Seiberg-Witten representation for Gaussian beta-ensemble,” arXiv:1103.5470 [hep-th].
- [46] I. P. Goulden and D. M. Jackson “Maps in Locally Orientable Surfaces and Integrals over Real Symmetric Surfaces,” *Can. J. Math.* Vol. 49(5), 1997 pp. 865-882
- [47] H. Ita, H. Nieder and Y. Oz, “Perturbative computation of glueball superpotentials for $SO(N)$ and $USp(N)$,” *JHEP* **0301** (2003) 018 [hep-th/0211261].
- [48] R. A. Janik and N. A. Obers, “ $SO(N)$ superpotential, Seiberg-Witten curves and loop equations,” *Phys. Lett. B* **553** (2003) 309 [hep-th/0212069].
- [49] H. Fuji and Y. Ookouchi, “Confining phase superpotentials for SO / Sp gauge theories via geometric transition,” *JHEP* **0302** (2003) 028 [arXiv:hep-th/0205301].
- [50] R. Dijkgraaf, S. Gukov, V. A. Kazakov and C. Vafa, “Perturbative analysis of gauged matrix models,” *Phys. Rev. D* **68** (2003) 045007 [arXiv:hep-th/0210238].
- [51] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic anomalies in topological field theories,” *Nucl. Phys. B* **405** (1993) 279 [arXiv:hep-th/9302103].